06/10 MATH 2230A

1. Integral for $w(t)$.

$$w(t) = u(t) + i\,v(t)$$

$u(t),\ v(t) \in \mathbb{R}$.

$w(t)$ & $\gamma(t) = u_1(t) + i\,v_1(t)$,

$a\,w(t) + b\,\gamma(t)$, $a, b \in \mathbb{C}$

$$\int a\,w(t) + b\,\gamma(t)$$

$$= a\int w(t) + b\int \gamma(t).$$

As $u(t)$ & $v(t)$ are also complex values ($\subset \mathbb{R}$),

$$\int w(t) = \int u(t) + i\int v(t).$$

$$\Rightarrow \begin{cases} \operatorname{Re}\left(\int w(t)\right) = \int \operatorname{Re}(w(t)) \\ \operatorname{Im}\left(\int (w(t))\right) = \int \operatorname{Im}(w(t)). \end{cases}$$

2. Chain Rule

f analytic, $\gamma(t) \in \mathbb{C}$.

$$\frac{d f(\gamma(t))}{dt}.$$

$f = U(z) + iV(z)$,

$f(\gamma(t)) = U(\gamma_x(t), \gamma_y(t)) + iV(\gamma_x(t), \gamma_y(t))$

$$\frac{d}{dt} f(\gamma(t)) = \frac{d}{dt} U(\underbrace{\gamma_x(t)}_{x}, \gamma_y(t))$$

x variable of U.

$$+ i \frac{d}{dt} V(\gamma_x(t), \gamma_y(t))$$

$$= \frac{dU}{dx}\Big|_{(\gamma_x(t), \gamma_y(t))} \gamma_x'(t)$$

$$+ \frac{dU}{dy}\Big|_{(\gamma_x(t), \gamma_y(t))} \gamma_y'(t)$$

$$+ i\left(V_x\big|_{(\gamma_x(t), \gamma_y(t))} \gamma_x'(t) + V_y\big|_{\sim} \gamma_y'(t) \right)$$

$$\frac{d}{dt} f(\gamma(t)) \overset{u_x = v_y}{\underset{u_y = -v_x}{=\!=\!=}} u_x|_\gamma \gamma_x'(t) - v_x|_\gamma \gamma_y'(t)$$

$$+ i\left(v_x|_\gamma \gamma_x'(t) + u_x|_\gamma \gamma_y'(t)\right)$$

$$= u_x|_\gamma \left(\gamma_x'(t) + i\gamma_y'(t)\right)$$

$$+ v_x|_\gamma \left(-\gamma_y'(t) + i\gamma_x'(t)\right)$$

$$= u_x|_\gamma \left(\gamma_x'(t) + i\gamma_y'(t)\right)$$

$$+ i v_x|_\gamma \left(\gamma_x'(t) + i\gamma_y'(t)\right)$$

$$= \underbrace{(u_x + i v_x)}_{f'|_\gamma} \underbrace{(\gamma_x'(t) + i\gamma_y'(t))}_{\gamma'(t).}$$

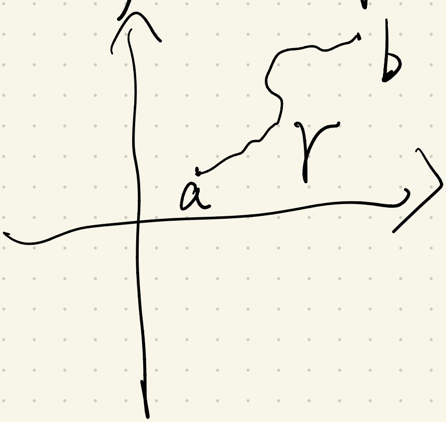
$$= f'(\gamma(t))\gamma'(t).$$

Chain Rule

: $[f(\gamma(t))]' = f'(\gamma(t))\,\gamma'(t)$.

f is analytic.

3. Integral of f analytic.



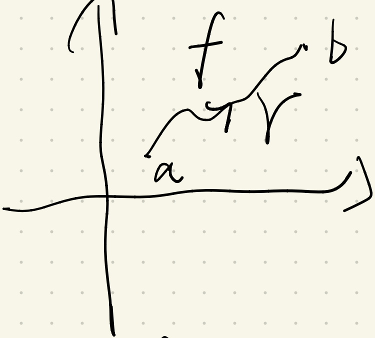
$$\underbrace{\int_\gamma f}_{\text{Integral makes sense (Finite).}}$$

$$\left|\int_\gamma f\right| \le \int |f|\,|d\gamma| \le \max_{z\in\gamma}|f|\,|\gamma|$$

Length of γ.

But γ is compact (Closed & Bdd), so f is continuous, $\Rightarrow |f|$ cts.

$\Rightarrow \max|f| < \infty$, $|\gamma| < \infty$. So this integral makes sense.



$\gamma : [0, 1] \to \mathbb{C}$.

also regular. $\begin{cases} \gamma(0) = a \\ \gamma(1) = b \end{cases}$.

$$\int_\gamma f(z)\, dz \overset{z = \gamma(t)}{=\!=\!=} \int_0^1 f(\gamma(t))\, \gamma'(t)\, dt. \quad (1)$$

(from a to b)

If different parametrize lead to different results? No.

$\varphi : [0, 1] \to \mathbb{C}$, $\varphi([0, 1]) = \gamma$.

$\varphi(0) = a$, $\varphi(1) = b$.

$g : [0, 1] \to [0, 1]$.

$g(0) = 0$, $g(1) = 1$, $\varphi(g(t)) = \gamma(t)$.

$\Rightarrow$ g is cts, g to monotone (strictly).

reparametrization.

(2) $\int_\gamma f = \int_0^1 f(\phi(t))\phi'(t)\,dt.$

$$\overset{t=g(s)}{=\!=} \int_0^1 f(\phi(g(s)))\,\phi'(g(s))\,g'(s)\,ds. \quad (2).$$

As $\phi(g(s)) = \gamma(s)$

Take $\Rightarrow$ derivative $\gamma'(s) = \phi'(g(s))\,g'(s)$

(2) $\rightarrow$ $\int_\gamma f = \int_0^1 f(\phi(t))\,\phi'(t)\,dt$

$$= \int_0^1 f(\gamma(s))\,\gamma'(s)\,ds$$

So parametrization not affect the value.

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Analycity $\Rightarrow$ C-R

If f is not fulfilling C-R

$\Rightarrow$ f is not Analytic.

C-R $\Leftrightarrow$ $U_x = V_y$ & $U_y = -V_x$.

$\neg$(C-R) $\Leftrightarrow$ $U_x \neq V_y$ or $U_y \neq -V_x$.